ON THE NUMBER OF CATENARIES THAT MAY BE DRAWN THROUGH TWO FIXED POINTS.*

By Dr. Harris Hancock, Chicago, Ill.

1. We shall see that there are three cases that come under our consideration :

    I. Two catenaries may be drawn through the two points ;

    II. Only one may be drawn through these points ;
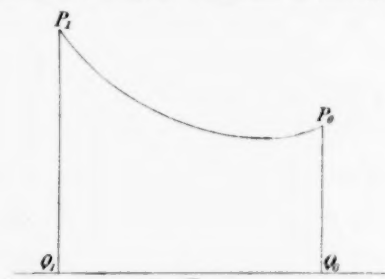
    III. No catenary can be drawn through the two points.

It was shown (Vol. X, p. 85) that

$$y = \tfrac{1}{2} m \left(e^{(x - x_0')/m} + e^{-(x - x_0')/m} \right), \tag{1}$$

and

$$\frac{x - x_0'}{m} = \log \text{nat} \left[\frac{y + \sqrt{y^2 - m^2}}{m} \right]. \tag{2}$$

Substitute in (2) first the coordinates of P_0 for x and y, and then those of P_1 ; subtract the latter result from the former, x_0' vanishes, and we have a transcendental equation in m, which we must investigate and find the roots.

FIG. I.

2. We assume that $y_1 \geqq y_0$. From the equation of the catenary

$$y_0 = \tfrac{1}{2} m \left(e^{(x_0 - x_0')/m} + e^{-(x_0 - x_0')/m} \right),$$

and

$$y_0^2 - m^2 = \tfrac{1}{4} m^2 \left(e^{(x_0 - x_0')/m} - e^{-(x_0 - x_0')/m} \right)^2.$$

Therefore

$$\sqrt{y_0{}^2 - m^2} = \pm \tfrac{1}{2} m \left(e^{(x_0 - x_0')/m} - e^{-(x_0 - x_0')/m}\right); \tag{I}$$

and from this relation it is seen that $\sqrt{y_0{}^2 - m^2}$ has a positive or a negative sign according as $x_0 - x_0' \gtrless 0$. Hence, also,

$$\frac{x_0 - x_0'}{m} = \pm \log \text{ nat} \left[\frac{y_0 + \sqrt{y_0{}^2 - m^2}}{m}\right]. \tag{a}$$

3. We must first show that such a figure as the one given cannot exist in the present discussion.
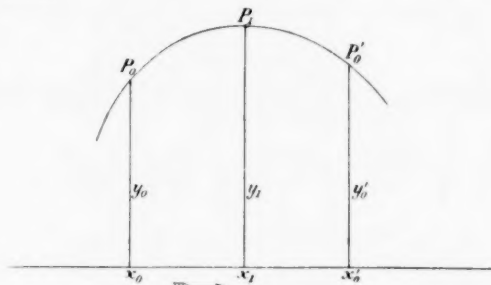
Fig. II.

Without regarding the figure we know that

$$y_1 = \tfrac{1}{2} m \left(e^{(x_1 - x_0')/m} + e^{-(x_1 - x_0)/m}\right), \quad x_1 - x_0' > 0.$$

That $x_1 - x_0'$ is necessarily positive is seen from the fact that the ordinate $y_0 = m$ corresponds to the value x_0' and is a minimum (p. 86).

Suppose next that $x_0' > x_1$.

By hypothesis $y_1 \geq y_0$, and further $m \leq y_0$, and consequently $m \leq y_1$. The form of the curve is then that given in the figure ; and we have within the interval x_0 to x_0' a value of x, for which the ordinate y is greater than it is at the end points.

y must therefore have within this interval a maximum value. But we have shown above (p. 86) that there is no maximum value of y.

Hence

$$\sqrt{y_1{}^2 - m^2} = + \tfrac{1}{2} m \left(e^{(x_1 - x_0')/m} - e^{-(x_1 - x_0')/m}\right),$$

and cannot have the *minus* sign as in equation (I).   Hence

$$\frac{x_1 - x_0'}{m} = + \log \text{ nat} \left[\frac{y_1 + \sqrt{y_1{}^2 - m^2}}{m}\right]. \tag{b}$$

4. Eliminate x_0' from (a) and (b) and noting that in (a) there is the $\pm$ sign, we have two different functions of m, which may be written :

$$f_1(m) = \log \text{nat} \left[\frac{y_1 + \sqrt{y_1^2 - m^2}}{m} \right] - \log \text{nat} \left[\frac{y_0 + \sqrt{y_0^2 - m^2}}{m} \right] - \frac{x_1 - x_0}{m},$$

and

$$f_2(m) = \log \text{nat} \left[\frac{y_1 + \sqrt{y_1^2 - m^2}}{m} \right] + \log \text{nat} \left[\frac{y_0 + \sqrt{y_0^2 - m^2}}{m} \right] - \frac{x_1 - x_0}{m},$$

two equations of a transcendental nature, which we have now to consider. We must see whether $f_1(m) = 0$, $f_2(m) = 0$ have roots with regard to m; that is, whether it is possible to give m positive real values, so that the equations $f_1(m) = 0$, $f_2(m) = 0$ will be satisfied : and if we can so determine m, we must then see whether the values of x_0' which may be then derived from equations (a) and (b) are one-valued.

The first derivative of $f_1(m)$ is

$$f_1'(m) = \frac{1}{m} \left\{ - \frac{y_1}{\sqrt{y_1^2 - m^2}} + \frac{y_0}{\sqrt{y_0^2 - m^2}} + \frac{x_1 - x_0}{m} \right\}. \tag{c}$$

On the right hand side of this expression $1/m$ is positive, also $(x_1 - x_0)/m$ is positive, and

$$\frac{1}{\sqrt{1 - m^2/y_0^2}} - \frac{1}{\sqrt{1 - m^2/y_1^2}} \quad \text{is positive, if } y_1 > y_0 ;$$

so that $f_1'(m)$ is positive in the interval $0 \ldots y_0$.

Further

$$f_1(0) = \log \text{nat} \, 2y_1 - \log \text{nat} \, (m = 0) - \log \text{nat} \, 2y_0 + \log \text{nat} \, (m = 0)$$

$$- (x_1 - x_0)/(m = 0) = - \infty .$$

5. It is further seen that $f_1(m)$ continuously increases within the interval $0 \ldots y_0$, so that $- \infty$ is the least value that $f_1(m)$ can take.

Again

$$f_1(y_0) = \log \text{nat} \left[\frac{y_1 + \sqrt{y_1^2 - y_0^2}}{y_0} \right] - \frac{x_1 - x_0}{y_0} . \tag{II}$$

Then if

  I. $f_1(y_0) < 0$, $f_1(m)$ has no root ;

  II. $f_1(y_0) = 0$, $f_1(m)$ has one root, viz., $m = y_0$;

  III. $f_1(y_0) > 0$, $f_1(m)$ has a root, $m_1 < y_0$.

When

$$f_1(y_0) < 0 , \quad P_1 \text{ is outside of the catenary ;}$$

$$f_1(y_0) = 0 , \quad P_1 \text{ is on the catenary ;}$$

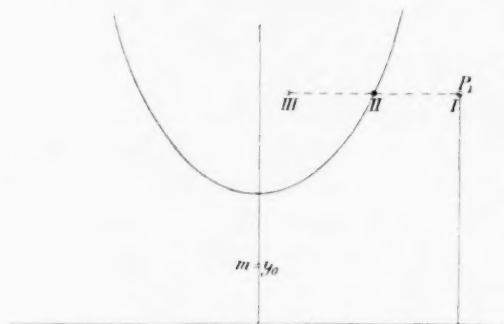$$f_1(y_0) > 0 , \quad P_1 \text{ is within the catenary.}$$

FIG. III.

This may be shown as follows :

$$y = \tfrac{1}{2} y_0 \left(e^{(x-x_0)/y_0} + e^{-(x-x_0)/y_0} \right),$$

since when $y = m$, $x = x_0'$; and, therefore, when $y = y_0 = m$, $x = x_0$.   We also have

$$y^2 - y_0^2 = \tfrac{1}{4} y_0^2 \left(e^{(x-x_0)/y_0} - e^{-(x-x_0)/y_0} \right)^2.$$

Hence

$$\sqrt{y^2 - y_0^2} = \pm \tfrac{1}{2} y_0 \left(e^{(x-x_0)/y_0} - e^{-(x-x_0)/y_0} \right) ;$$

the positive sign to be taken when $x > x_0$, and the negative sign, when $x < x_0$.

We also have

$$x - x_0 = y_0 \log \text{nat} \left[\frac{y + \sqrt{y^2 - y_0^2}}{y_0} \right].$$

Comparing this equation with equation (II) above, and noticing the figure, it is seen that, when

$$x_1 - x_0 = y_0 \log \text{nat} \left[\frac{y_1 + \sqrt{y_1^2 - y_0^2}}{y_0} \right], \text{ then } P_1 \text{ is on catenary,}$$

$$x_1 - x_0 > y_0 \log \text{nat} \left[\frac{y_1 + \sqrt{y_1^2 - y_0^2}}{y_0} \right], \text{ then } P_1 \text{ is outside catenary,}$$

$$x_1 - x_0 < y_0 \log \text{nat} \left[\frac{y_1 + \sqrt{y_1^2 - y_0^2}}{y_0} \right], \text{ then } P_1 \text{ is within catenary.}$$

Hence, when $f_1(y_0) > 0$, there is one and only one real root in the interval

$0 \ldots y_0$, and we can draw through the points P_1' and P_0' a catenary, for which the abscissa of the lowest point is $< x_0$.

6. *The Discussion of $f_2(m) = 0$.* We saw (Art. 4) that

$$f_2(m) = \log\left[\frac{y_1 + \sqrt{y_1^2 - m^2}}{m}\right] + \log\left[\frac{y_0 + \sqrt{y_0^2 - m^2}}{m}\right] - \frac{x_1 - x_0}{m}.$$

Therefore

$$f_2'(m) = -\frac{1}{m^2}\left\{\frac{y_1 m}{\sqrt{y_1^2 - m^2}} + \frac{y_0 m}{\sqrt{y_0^2 - m^2}} - (x_1 - x_0)\right\}.$$

When m changes from 0 to y_0, $\sqrt{y_0^2/m^2 - 1}$ continuously decreases, and consequently $y_0/\sqrt{y_0^2/m^2 - 1}$ becomes greater and greater. Hence if the expression $-m^2 f_2'(m)$ takes the value 0, it takes it only once in the interval from 0 to y_0.

That this expression does take the value 0 within this interval is seen from the fact that, for $m = 0$, $-m^2 f_2'(0) = -(x_1 - x_0)$, where $x_1 - x_0 > 0$, so that $-m^2 f_2'(0)$ has a negative value; but, for $m = y_0$, $-m^2 f_2'(0) = +\infty$, so that the expression must take the value zero between these two values of m.

Let μ be this value of m which satisfies the equation, so that

$$\frac{y_1 \mu}{\sqrt{y_1^2 - \mu^2}} + \frac{y_0 \mu}{\sqrt{y_0^2 - \mu^2}} - (x_1 - x_0) = 0, \tag{A}$$

which is an algebraical equation of the 8th degree in μ; or, an algebraical equation of the 4th degree in μ^2.

7. *An approximate geometrical construction for the root that lies between* 0 *and* y_0. In the figure it is seen that the triangles $P_0 Q_0 A_0$ and $P_0 Q_0 C_0$ are
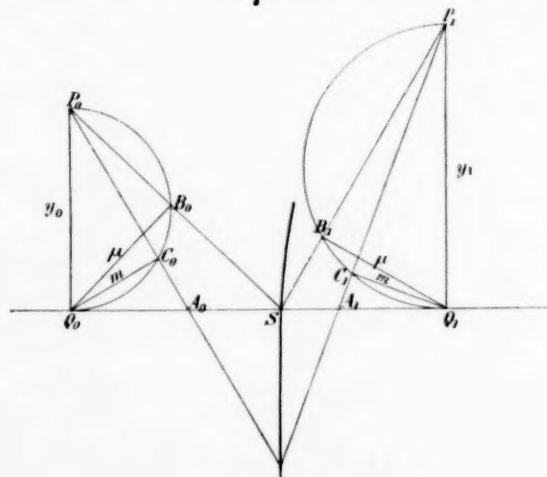

Fig. IV.

similar, as are also the triangles $P_1Q_1A_1$ and $P_1Q_1C_1$. Hence, if m is the length of the line $Q_0C_0 = Q_1C_1$, we have

$$Q_0A_0 = \frac{y_0 m}{\sqrt{y_0^2 - m^2}}, \quad \text{and} \quad Q_1A_1 = \frac{y_1 m}{\sqrt{y_1^2 - m^2}}.$$

By taking equal lengths $Q_0C_0 = Q_1C_1$ on the two semicircles and prolonging P_0C_0 and P_1C_1 until they intersect, we have as the locus of the intersections a certain curve. This curve must intersect the axis of the x in a point S, say.

Noting that

$$Q_0S + Q_1S = Q_0Q_1 = x_1 - x_0,$$

it follows that

$$\frac{y_0 \cdot Q_0B_0}{\sqrt{y_0^2 - Q_0B_0^2}} + \frac{y_1 \cdot Q_1B_1}{\sqrt{y_1^2 - Q_1B_1^2}} = x_1 - x_0,$$

which compared with the equation (A) above, shows that

$$Q_0B_0 = Q_1B_1 = \mu.$$

8. *The curves represented by the equations $f_1(m)$ and $f_2(m)$.*

Equation (c) gives $f_1'(y_0) = \infty$; that is, the tangent to the curve $f_1(m)$ at the point y_0 is parallel to the axis of y.

FIG. V.

Further $f_1(0) = -\infty$, so that the negative half of the axis of y is asymptotic to the curve $f_1(m) = 0$. The branch of the curve is here algebraic, since $f_1(m) = 0$ for $m = 0$ is purely algebraically infinite.

9. *Consider next the curve $f_2(m) = 0$.*

It is seen that $f_1(y_0) = f_2(y_0)$; and also $f_2'(y_0) = -\infty$, so that the tangent at this point is also parallel to the axis of the y. Further the negative half of the axis of the y is an asymptote to the curve; but the branch of $f_2(m) = 0$ is transcendent at the point $m = 0$; because logarithms enter in the development of this function in the neighborhood of $m = 0$, as may be seen as follows:
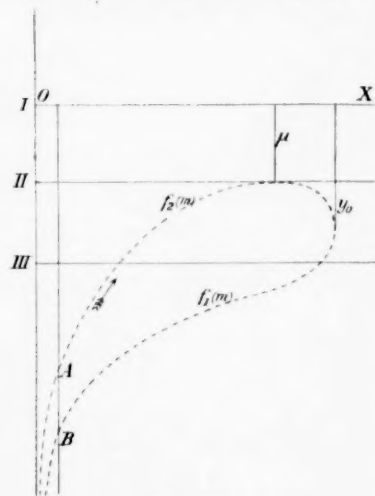
$$f_2(m) = \log\left[\frac{y_1 + \sqrt{y_1^2 - m^2}}{m}\right] + \log\left[\frac{y_0 + \sqrt{y_0^2 - m^2}}{m}\right] - \left[\frac{x_1 - x_0}{m}\right]$$

$$= -\left[\frac{x_1 - x_0}{m}\right] - 2\log m + P(m),$$

where $P(m)$ denotes a power series in positive and integral ascending powers of m. Hence the function behaves in the neighborhood of $m = 0$ as a logarithm.

10. We saw that

$$f_2'(m) = -\frac{1}{m^2}\left[\frac{y_1 m}{\sqrt{y_1^2 - m^2}} + \frac{y_0 m}{\sqrt{y_0^2 - m^2}} - (x_1 - x_0)\right].$$

For the value $m = \mu$, the expression within the brackets is zero, and when $m = 0$, this expression becomes $-(x_1 - x_0)$, and is negative. As seen above in the interval $m = 0$ to $m = y_0$, the expression

$$\left[\frac{y_1 m}{\sqrt{y_1^2 - m^2}} + \frac{y_0 m}{\sqrt{y_0^2 - m^2}} - (x_1 - x_0)\right]$$

becomes greater and greater, so that between the values $m = \mu$ and $m = y_0$, it is positive ; and between the values $m = 0$ and $m = \mu$, it is negative.

Furthermore $f_2'(m)$ is positive between $m = 0$ and $m = \mu$, and negative between $m = \mu$ and $m = y_0$.

Hence $f_2(m)$ increases between $m = 0$ and $m = \mu$, and decreases between $m = \mu$ and $m = y_0$; and consequently $f_2(\mu)$ is a maximum.
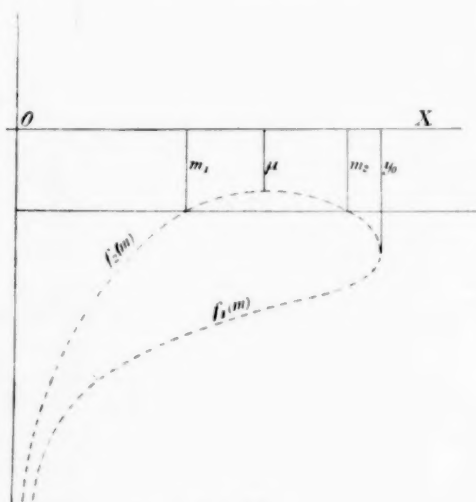

Fig. VI.

11. We must consider the function $f_2(m)$ when m is given different values, and see how many catenaries may be laid between the points P_0 and P_1.
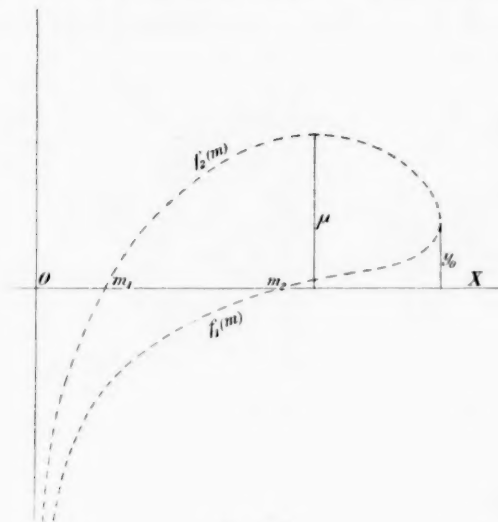
FIG. VII.

We have :

CASE I. $f_2'(\mu) < 0$.

In this case $f_2'(m)$ is nowhere zero, and there is no root of $f_2(m)$ which we can use.

There is also no root of $f_1(m)$, since

$$f_2'(y_0) < 0 \text{ and } f_2'(y_0) = f_1'(y_0)$$

so that $f_1'(y_0) < 0$, and there is no root (see Art. 5).

CASE II. $f_2'(\mu) = 0$.

All values of m other than μ cause $f_2'(m)$ to be negative, so that there is one and only one root of the equation $f_2(m) = 0$, and consequently only one catenary.

In this case $f_1(m)$ can never be zero ; since $f_2'(y_0) < 0$, and $f_1'(y_0) = f_2'(y_0)$, so that $f_1'(y_0) < 0$, with the result similar to that in Case I.

CASE III. $f_2'(\mu) > 0$.

We have here two catenaries.   One root of $f_2(m) = 0$ lies between 0 and μ, and *often* another between μ and y_0, as we may see as follows :—

$$f_2'(+ 0) = - \infty \text{ and } f_2'(\mu) > 0 .$$

Since $f_2'(m)$ continuously increases in the interval $+ 0 \ldots \mu$, it can take the value 0 only once within this interval.

In the interval $\mu \ldots y_0$, $f_2'(m)$ continuously decreases, so that if $f_2'(y_0) > 0$, there is no root of $f_2'(m) = 0$ within this interval; but if $f_2'(y_0) \gtrless 0$, then there

is one and only one root within this interval, and in the latter case there are two catenaries.

We must next consider the roots of $f_1(m)$. When $f_2(y_0) < 0$, then is $f_1(y_0) < 0$, so that there is no root of $f_1(m) = 0$. But when $f_2(y_0) = 0$, then $f_1(y_0) = 0$; and $f_1(m) = 0$ has the root $m = y_0$ which was given above.

Therefore :

A) $\left\{ \begin{array}{l} \text{When } f_2(y_0) < 0, f_2(m) \text{ has two roots; and when } f_2(y_0) = 0, f_2(m) \text{ has} \\ \text{a root in addition to the root which belongs to } f_2(y_0) = f_1(y_0). \end{array} \right.$

B) $\left\{ \begin{array}{l} \text{But when } f_2(y_0) > 0, \text{ then there is only one root for } f_2(m) = 0, \text{ which} \\ \text{lies between } 0 \ldots \mu; \text{ this root is denoted by } m_1. \end{array} \right.$

12. From the formulæ (Art. 4) for $f_1(m)$ and $f_2(m)$, we have

$$f_2(m) = f_1(m) + 2 \log \left[\frac{y_0 + \sqrt{y_0^2 - m^2}}{m} \right].$$

We consider m within the interval $0 \ldots y_0$; for

$$m = 0, \quad \frac{y_0 + \sqrt{y_0^2 - m^2}}{m} = \infty ;$$

and for

$$m = y_0, \quad \frac{y_0 + \sqrt{y_0^2 - m^2}}{m} = 1 .$$

Consequently within this interval $\log \left[\dfrac{y_0 + \sqrt{y_0^2 - m^2}}{m} \right]$ is positive, and therefore also $f_2(m) > f_1(m)$; and since $f_2(m_1) = 0$, it follows that $f_1(m_1) < 0$.

On the other hand $f_1(y_0) = f_2(y_0)$; and since $f_2(y_0) > 0$, we have $f_1(y_0) > 0$.

Moreover, within the interval $0 \ldots y_0$, $f_1(m)$ continuously increases, and $f_1(+0) < 0$, so that within the interval $0 \ldots m_1$, $f_1(m)$ has no root, and within the interval $m_1 \ldots y_0$ one root.

Hence under B) $f_2(m)$ has a root m_1 within the interval $0 \ldots \mu$, and only one root, and $f_1(m)$ has a root between m_1 and y_0 and only one ; making a total under the heading B) of two catenaries.

We have the following summary :—

1°. $f_2(m) < 0$, no catenary ;

2°. $f_2(m) = 0$, one catenary ;

3°. $f_2(m) > 0$, two catenaries.

13. *On the consideration of the intersection of the tangents drawn to the catenary at the points P_0 and P_1.*

CASE I. As shown above there is here no catenary, so that the consideration of the tangents is without interest.

CASE II. $f_2(\mu) = 0.$

Here the catenary enjoys the remarkable property that the tangents drawn at the points P_0 and P_1 intersect on the x-axis.   In order to show this we must
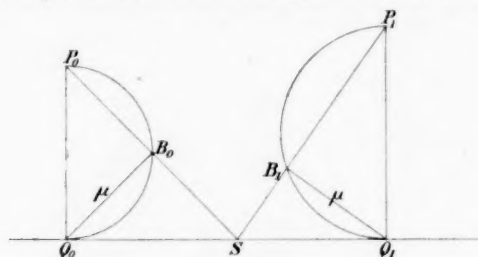

Fig. VIII.

go back to the construction of the tangents at the points P_0 and P_1. It was seen (p. 87) that points B_0 and B_1 were found on the semi-circumferences $P_0 B_0 Q_0$ and $P_1 B_1 Q_1$ such that $Q_0 B_0 = Q_1 B_1$ ($m = \mu$ in this case); and that then the lines $P_0 B_0$ and $P_1 B_1$ were the required tangents, which intersect on the x-axis (see Art. 7).

CASE III. $f_2'(\mu) > 0$.

A) $f_2'(y_0) \gtrless 0$.
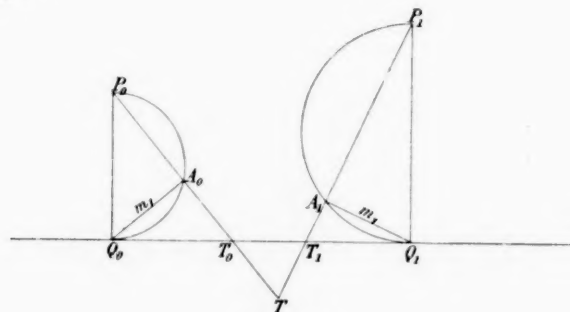
Fig. IX.

Then, as already shown, $f_2(m) = 0$ has two roots, one of which lies between 0 and μ, and the other between μ and y_0.

Let these roots be m_1 and m_2 respectively.   For the root m_1, we have

$$ Q_0 T_0 = \frac{y_0 m_1}{\sqrt{y_1^2 - m_1^2}} ; \quad Q_1 T_1 = \frac{y_1 m_1}{\sqrt{y_1^2 - m_1^2}} . $$

We assert that here the intersection of the tangents at P_0 and P_1 lies on the other side of the x-axis.

In order to show this we need only prove that

$$ Q_0 T_0 + Q_1 T_1 < Q_0 Q_1 . $$

This is shown as follows :—

$$f_2''(m_1) = -\frac{1}{m_1^2}\left\{\frac{y_1 m_1}{\sqrt{y_1^2 - m_1^2}} + \frac{y_0 m_1^2}{\sqrt{y_0^2 - m_1^2}} - (x_1 - x_0)\right\}.$$

Now since $f_2''(m)$ within the interval $0 \ldots \mu$ is positive, and since m_1 lies within this interval, it follows that $f_2''(m_1)$ is positive.

Therefore $-(m_1)^2 f_2''(m_1)$ is negative, and consequently $Q_0 T_0 - Q_1 T_1 - Q_0 Q_1$ is negative.

REMARK. In this consideration the whole interpretation depends upon the fact that the root lies in the interval $0 \ldots \mu$, and the same discussion is applicable to case B) where $f_2(y_0) > 0$, and where the root lies between $0 \ldots \mu$.

15. *On the consideration of the root* m_2.

1°. When $f_2(y_0) \lessgtr 0$.

The root lies within the interval $\mu \ldots y_0$. $f_2''(m)$ is negative within this interval ; therefore $-m^2 f_2''(m)$ is positive, and consequently

$$\frac{y_1 m^2}{\sqrt{y_1^2 - m_2^2}} + \frac{y_0 m^2}{\sqrt{y_0^2 - m_2^2}} - (x_1 - x_0) > 0 ;$$

$$\therefore \quad Q_0 T_0 + Q_1 T_1 > Q_0 T_0 ;$$
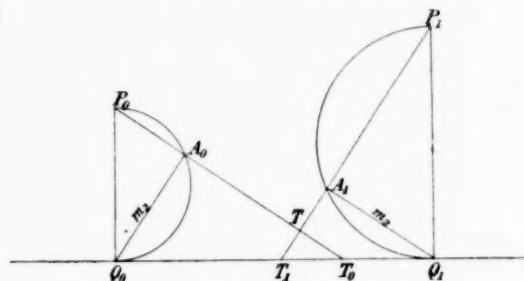
FIG. X.

so that T is on the same side of the x-axis as the curve.

2°. When $f_2(y_0) > 0$; then the root m_2 is a root of the equation $f_1(m) = 0$, so that we have here to consider the sign of

$$-\frac{y_1 m_2}{\sqrt{y_1^2 - m_2^2}} + \frac{y_1 m_2}{\sqrt{y_0^2 - m_2^2}} - (x_1 - x_0)$$

within the interval $0 \ldots y_0$.

We have proved that within this interval $f_1''(m)$ is positive, and since

$$f_1''(m_2) = -\frac{1}{m_2^2}\left[\frac{y_1 m_2}{\sqrt{y_1^2 - m_2^2}} - \frac{y_0 m_2}{\sqrt{y_0^2 - m_2^2}} - (x_1 - x_0)\right]$$

is positive, it follows that

$$\left[\frac{y_1 m_2}{\sqrt{y_1^2 - m_2^2}} - \frac{y_0 m_2}{\sqrt{y_0^2 - m_2^2}} - (x_1 - x_0) \right]$$

is negative.   Hence

$$\frac{y_1 m_2}{\sqrt{y_1^2 - m_2^2}} - \frac{y_0 m_2}{\sqrt{y_0^2 - m_2^2}} < (x_1 - x_0) .$$

And consequently

$$\frac{y_0 m_2}{\sqrt{y_0^2 - m_2^2}} - \frac{y_1 m_2}{\sqrt{y_1^2 - m_2^2}} > (x_1 - x_0) .$$

Since $\dfrac{y_1 m_2}{\sqrt{y_1^2 - m_2^2}}$ is a positive quantity, it follows *a fortiori* that

$$\frac{y_1 m_2}{\sqrt{y_1^2 - m_2^2}} + \frac{y_0 m_2}{\sqrt{y_0^2 - m_2^2}} > (x_1 - x_0) ,$$

and the intersection lies on the same side of the x-axis as the curve.

16. *Lindelöf's Theorem* (1860).

If we suppose the catenary to revolve around the x-axis, as also the lines $P_0 T$ and $P_1 T$, then the surface generated by the revolution of the catenary is
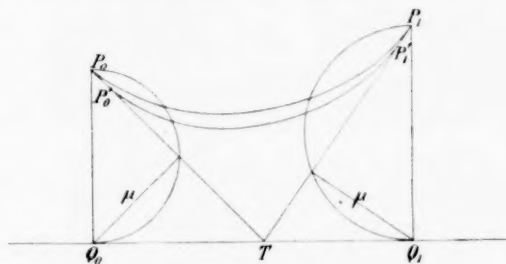
FIG. XI.

equal to the sum of the surfaces generated by the revolution of the two lines $P_0 T$ and $P_1 T$ about the x-axis.

Suppose that with T as *center of similarity* (Aehnlichkeitspunkt), the curve $P_0 P_1$ is subjected to a strain so that P_0 goes into the point P_0', and P_1 into the point P_1', the distance $P_0 P_0'$ being very small and equal, say, to $a = P_1 P_1'$. Then

$$P_0 T : P_0' T = 1 : 1 - a .$$

For the sake of abbreviation, let

M_0 denote the surface generated by $P_0 T$, M_0' that generated by $P_0' T$,
M_1 "   "   "   "   " $P_1 T$, M_1' "   "   " $P_1' T$,
S that by the catenary $P_0 P_1$, S' that by the catenary $P_0' P_1'$.

From the nature of the strain the tangents $P_0 T$ and $P_1 T$ are tangents to the new curve at the points P_0' and P_1', so that we may consider $P_0 P_0' P_1' P_1$ as a variation of the curve $P_0 P_1$.

It is seen that

$$S : S' = 1 : (1 - a)^2 ;$$

$$M_0 : M_0' = 1 : (1 - a)^2 ;$$

$$M_1 : M_1' = 1 : (1 - a)^2 .$$

Now from the figure we have as the surface of rotation of $P_0 P_0' P_1' P_1$

$$(M_0 - M_0') + S' + (M_1 - M_1') + ((a^2)) = S,$$

where $((a^2))$ denotes a variation of the second order.

Therefore

$$S - S' = (M_0 - M_0') + (M_1 - M_1') + ((a^2)) .$$

Hence

$$S [1 - (1 - a)^2] = M_0 [1 - (1 - a)^2] + M_1 [1 - (1 - a)^2] + ((a^2)),$$

and consequently

$$2aS = 2a M_0 + 2a M_1 + ((a^2)),$$

or finally

$$S = M_0 + M_1 ,$$

a result which is correct to a differential of the first order.

In a similar manner

$$S' = M_0' + M_1' ;$$

so that

$$S - S' = (M_0 - M_0') + (M_1 - M_1') ,$$

or

$$S = (M_0 - M_0') + S' + (M_1 - M_1')$$

is an expression which is absolutely correct.

17. *Geometrical Proof.*

We have seen that

$$\frac{y_0 \mu}{\sqrt{y_0^2 - \mu^2}} + \frac{y_1 \mu}{\sqrt{y_1^2 - \mu^2}} - (x_1 - x_0) = 0 , \tag{1}$$

and (see Fig. IV, Art. 7)

$$P_0 S = \frac{y_0^2}{\sqrt{y_0^2 - \mu^2}} ; \quad P_1 S = \frac{y_1^2}{\sqrt{y_1^2 - \mu^2}} . \tag{2}$$

The surfaces of the two cones are, therefore, equal to

$$\frac{y_0 \cdot y_0^2 \pi}{\sqrt{y_0^2 - \mu^2}}, \text{ and } \frac{y_1 \cdot y_1^2 \pi}{\sqrt{y_1^2 - \mu^2}} .$$

The surface generated by the catenary is

$$\int_{x_0}^{x_1} 2y\pi ds\,.$$

In the catenary $ds = \dfrac{y}{m}\,.\,dx$ (see p. 87), so that

$$\int_{x_0}^{x_1} 2y\pi ds = \int_{x_0}^{x_1}\frac{2y^2\pi dx}{m} = \pi\,.\,2\int_{x_0}^{x_1}\frac{m^2}{4}[e^{2(x-x_0')/m} + 2 + e^{-2(x-x_0')/m}]\,dx/m$$

$$= \tfrac{1}{4}\pi m^2\left[e^{2(x-x_0')/m} - e^{-2(x-x_0')/m} + 4x/m\right]_{x_0}^{x_1}$$

$$= \pi\left[\tfrac{1}{2}m\left(e^{(x-x_0')/m} + e^{-(x-x_0')/m}\right).\tfrac{1}{2}m\left(e^{(x-x_0')/m} - e^{-(x-x_0')/m}\right) + mx\right]_{x_0}^{x_1} \quad \text{(A)}$$

$$= \pi\left[\pm y\sqrt{y^2 - m^2} + mx\right]_{x_0}^{x_1}$$

$$= \pi\left[y_1\sqrt{y_1{}^2 - m^2} + y_0\sqrt{y_0{}^2 - m^2} + m(x_1 - x_0)\right], \quad \text{(B)}$$

where we have taken the $+$ sign with $y_0\sqrt{y_0{}^2 - m^2}$ because $x_0 - x_0'$ is negative, hence $e^{(x-x_0')/m} - e^{-(x-x_0')/m}$ in (A) is negative.   But from (1)

$$x_1 - x_0 = \frac{y_1\mu}{\sqrt{y_1{}^2 - \mu^2}} + \frac{y_0\mu}{\sqrt{y_0{}^2 - \mu^2}}.$$

Substituting in (B), we have, after making $m = \mu$, for the area generated by the revolution of the catenary

$$\pi\left[y_1\sqrt{y_1{}^2 - \mu^2} + \frac{y_1\mu^2}{\sqrt{y_1{}^2 - \mu^2}} + y_0\sqrt{y_0{}^2 - \mu^2} + \frac{y_0\mu^2}{\sqrt{y_0{}^2 - \mu^2}}\right]$$

$$= \pi\left[\frac{y_1{}^3}{\sqrt{y_1{}^2 - \mu^2}} + \frac{y_0{}^3}{\sqrt{y_0{}^2 - \mu^2}}\right],$$

which as shown above is the sum of the surface areas of the two cones.

18. Let us consider again for a moment Fig. XII, in which the strain is represented.   In order to have a minimum surface of revolution the curve which we rotate must satisfy the differential equation of the problem.   If then we had a minimum, this would be brought about by the rotation of the catenary, for the catenary is the curve which satisfies the differential equation; but in our figure this curve can produce no minimal surface of revolution for two reasons : 1° because, drawing tangents (it is shown later that an infinite number may be drawn) which intersect on the x-axis, it is seen that the rotation of $P_0'P_1'$ is the same as that of the two lines $P_0'T$ and $P_1'T$, as shown above ; so that there are an infinite number of lines that may be drawn between

P_0 and P_1 which give the same surface of revolution as the catenary between these points ; 2° because between P_0 and P_1 lines may be drawn which when caused to revolve about the x-axis would produce a smaller surface area than
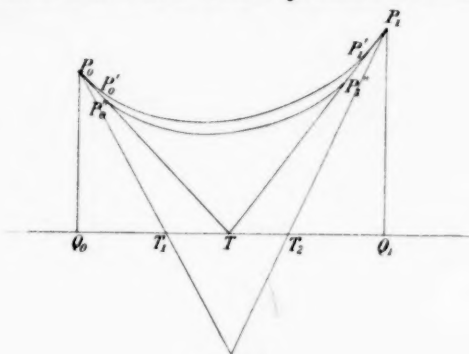

Fig. XII.

that produced by the revolution of the catenary. For the surface generated by the revolution of $P_0'P_1'$ is the same as that generated by $P_0'P_0''P_1''P_1'$. But the straight lines $P_0'P_0''$ and $P_1'P_1''$ do not satisfy the differential equation of the problem, since they are not catenaries. Hence the first variation along these lines is $\gtrless 0$, so that between the points P_0', P_0'' and P_1', P_1'' curves may be drawn whose surface of rotation is smaller than that generated by the straight lines $P_0'P_0''$ and $P_1'P_1''$.

The Case II given above and known as the transition case, i. e., where the point of intersection of the tangents pass from one side to the other side of the axis of the x, affords also no minimal surface, since, as seen above, there are, by varying the variable a, an infinite number of equal surfaces of revolution.

19. In Case III we had two roots of m, which we called m_1 and m_2, where $m_2 > m_1$. We consider first the catenary with parameter m_1. This parameter satisfies the inequality

$$\frac{y_1 m_1}{\sqrt{y_1^2 - m_1^2}} + \frac{y_0 m_1}{\sqrt{y_0^2 - m_1^2}} < x_1 - x_0. \qquad (A)$$

The equation of the tangent to the curve is

$$\frac{dy}{dx} = \frac{y' - y}{x' - x},$$

where x' and y' are the running coordinates. The intersection of this line with the axis of x is

$$x' - x = -\frac{y}{\frac{dy}{dx}}, \text{ or } x' = x - \frac{y}{\frac{dy}{dx}}; \text{ i. e. } x' = x - m\left[\frac{e^{(x-x_0')/m} + e^{-(x-x_0')/m}}{e^{(x-x_0')/m} - e^{-(x-x_0')/m}}\right].$$

Hence when $x = x_0'$, $x' = -\infty$, and when $x = +\infty$, $x' = +\infty$.

On the other hand dx'/dx is always positive, so that x' always increases when x increases, and the tangent passes from $-\infty$ along the x-axis to $+\infty$, and never passes twice through the same point.

20. It remains yet to show that there are portions of the arc of the catenary, which may be taken between the points P_0 and P_1 in such a way that tangents drawn to the catenary at the end points of these portions of arc intersect on the x-axis.

Let us suppose the point P_0 fixed, while the point P_1' moves from the lower point of the catenary in the direction P_1; the tangent to the catenary at this point moves from $-\infty$ along the x-axis until it reaches the point T_2 (Fig. XII), and then P_1' is at the point P_1, so that the tangent must have passed through the point T_1.

It is thus seen that there are an infinite number of pairs of points on the catenary between the points P_0 and P_1, such that the tangents at any of these pairs of points intersect on the x-axis; and as in Case III, there can be no minimum. Such pairs of points are known as *conjugate points*. Since x' always increases with x, no portion of the x-axis is covered twice by the intersection of the tangents.

Case III, when $m = m_2$; then there is a minimum as Prof. Weierstrass showed in the year 1879. He proved that within a certain region the curves
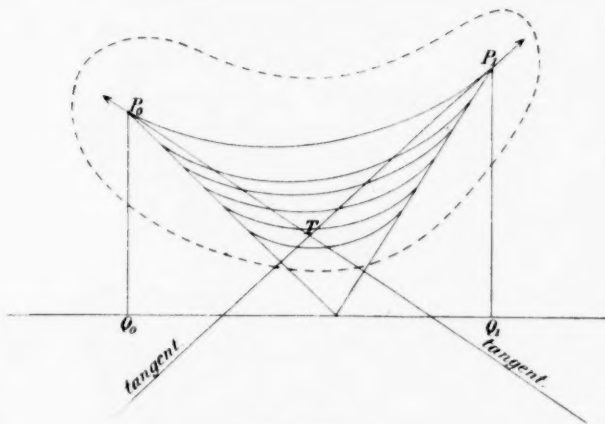

Fig. XIII.

that are caused to exist the one from the other by strains (see Art. 16) having a common point on the x-axis as center of similarity do not cut each other; and from this results the existence of a minimum, as will be seen later.

A GENERAL THEOREM RELATING TO TRANSVERSALS, AND ITS CONSEQUENCES.

By Mr. A. L. Candy, Lincoln, Nebraska.

Through the extremities of two fixed chords of a given circle two intersecting lines are drawn, and upon the two fixed chords circles are described passing through the intersection of this second pair of lines. Also, any right line is drawn through this same point of intersection. It is required to find the relation of the segments intercepted upon this transversal by the three circles and the two fixed chords ; all segments being measured from the intersection of the second pair of lines.

Consider first the four segments determined by the given circle and the two fixed chords, under the following cases :*—

CASE I. When the point of intersection is within the given circle and the transversal cuts the fixed chords within the given circle.

Let CD and $C'D'$ be the two fixed chords, and let O be the intersection of $C'D$ and CD'. Let any transversal, drawn through O, intersect the given circle in A and B, CD in E_1, and $C'D'$ in E_2, then OA, OB, OE_1, and OE_2 are the segments to be considered.

Let A_1, A_2, A_3, $A_4 \equiv$ the areas of the triangles E_1OC, E_2OC', E_1OD, and E_2OD', respectively.

Let $OB = m$, $OA = n$, $OE_2 = p$, $OE_1 = q$, $OC' = a$, $OD = b$, $OC = c$, $OD' = d$, $E_1C = x$, $E_1D = y$, $E_2C' = v$, and $E_2D' = z$.

Then,

$$\frac{A_1}{A_2} = \frac{cx}{av}, \quad (1) \qquad \frac{A_3}{A_4} = \frac{by}{dz}, \quad (2)$$

$$\frac{A_1}{A_4} = \frac{cq}{dp}, \quad (3) \qquad \frac{A_3}{A_2} = \frac{bq}{ap}, \quad (4)$$
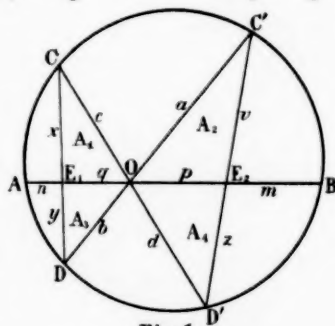

Fig. 1.

since $\angle C = \angle C'$, $\angle D = \angle D'$, etc.;

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* In all cases corresponding points will be designated by the same letters and the same notation will be used, so that it need not be repeated.

$$\therefore \qquad \frac{A_1 A_3}{A_2 A_4} = \frac{bcq^2}{adp^2} = \frac{bcxy}{advz}, \tag{5}$$

whence

$$\frac{q^2}{p^2} = \frac{xy}{vz} = \frac{A E_1 \cdot E_1 B}{A E_2 \cdot E_2 B} = \frac{(n-q)(m+q)}{(n+p)(m-p)}, \tag{6}$$

$$\frac{q^2}{p^2} = \frac{mn - q(m-n) - q^2}{mn + p(m-n) - p^2}; \tag{7}$$

$$\therefore \qquad mn(p-q) = pq(m-n), \tag{8}$$

by simplification, or
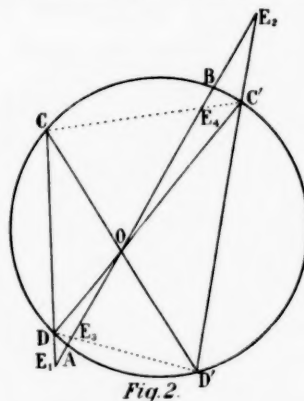
$$\frac{mn}{pq} = \frac{m-n}{p-q}. \tag{9}$$

∴ The products of the segments intercepted by the circle and the two chords, respectively, are proportional to their differences (or sums).

Since, by hypothesis, AB is any chord through O, it can be so drawn that O shall be its middle point. Then, $m - n = 0$, and from (8) $mn(p-q) = 0$; and therefore, since $mn \gtrless 0$,

$$p = q.^* \tag{10}$$

Case II. When O is within the given circle and E_1 and E_2 are without.

Fig. 2.

The angles C and C', D and D' are now supplementary, and equations (1) ... (5) are still satisfied, but (6) becomes

$$\frac{q^2}{p^2} = \frac{(q-n)(m+q)}{(n+p)(p-m)}, \tag{11}$$

which again reduces to (8), since the signs of two factors are changed in the second fraction.

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* See Stewart's Geom., p. 288, Ex. 29.   It was this exercise that suggested this investigation ; and all the following results will be found to be consequences of this simple relation in (8).

. Next draw CC' and DD' intersecting AB in E_4 and E_3, respectively; let $OE_4 = r$, $OE_3 = s$; then, as in Case I,

$$mn\,(r - s) = rs\,(m - n)\,; \tag{12}$$

$\therefore$

$$\frac{p - q}{r - s} = \frac{pq}{rs}\,, \tag{13}$$

from (8) and (12).

$\therefore$ The segments intercepted by opposite pairs of chords joining the same points, possess the same relation as the segments made by the circle and one pair of opposite chords, and when any pair are equal all three pairs are equal.

When AB meets CD and $C'D'$ (produced) in the same direction from O equation (8) becomes

$$mn\,(p + q) = pq\,(m - n)\,, \tag{14}$$

which is just as it should be if we agree that direction shall determine the algebraic sign, for then p and q have the same direction and must be regarded as having the same signs.

CASE III. When O is without the circle and E_1 and E_2 within.

The proof in this case is so similar to the first two that it is not necessary to produce it. However, it can be seen from Fig. 3 that m, n, p and q all extend in the same direction from O, and therefore m and n as well as p and
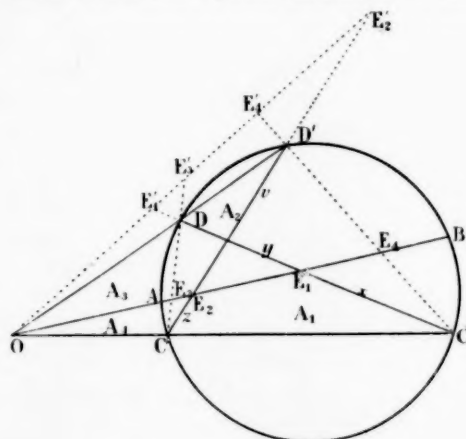

Fig. 3.

q must be regarded as having the same sign, in order to make the preceding general statement agree with the result of the proof which gives

$$mn\,(p + q) = pq\,(m + n)\,. \tag{15}$$

CASE IV. When O is without the circle, and CD, $C'D'$ intersect within the circle.

Using the same triangles as in Case I, it is easily seen that

$$\frac{A_1 A_3}{A_2 A_4} = \frac{bcq^2}{adp^2} = \frac{bcxy}{advz}, \tag{16}$$

since $\angle C = \angle C'$, $\angle D = \angle D'$, and angles at O are common.

$$\therefore \qquad \frac{q^2}{p^2} = \frac{xy}{vz} = \frac{(q-n)(m-q)}{(p-n)(m-p)}, \tag{17}$$

which gives as in Case III

$$mn(p+q) = pq(m+n). \tag{18}$$

Now let $C'D$ and CD' be drawn intersecting AB in E_3 and E_4 respectively; let $OE_4 = r$, $OE_3 = s$; then substituting in (18) gives, by Case III,

$$mn(r+s) = rs(m+n); \tag{19}$$

and dividing (18) by (19) gives

$$\frac{p+q}{r+s} = \frac{pq}{rs}, \tag{20}$$

an equation that is independent of m and n.

This result is analogous to (13) but the interpretation is more inclusive. It shows that the relation between the segments formed by the two pairs of lines remains the same when the transversal does *not* intersect the given circle, as shown by $OE_1'E_3'$, etc., Fig. 3. We shall have occasion to refer to this equation again. It is well to notice in passing that when O is on the given circle, either m and p or n and q both vanish, and (8) and (20) are verified.

It has now been shown that, having proper regard to signs, equations (8), (13), and (20) are true under all possible conditions, i. e. they are identical.

Next in order to determine the relation between the segments intercepted by the given circle and the two circles described upon the two fixed chords CD, $C'D'$, and passing through O, the process of inversion will be used.

Inverting Fig. 1, using O as the centre of inversion gives Fig. 4, in which the inverse points are designated by the same letters; the lines CD and $C'D'$ inverting into the circles COD and $C'OD'$, respectively, lines through O remaining unchanged.

Let $OB = m'$, $OA = n'$, $OE_2 = p'$, $OE_1 = q'$; and compare these with m, n, p, q of Fig. 1; then

$$m = \frac{1}{m'}, \quad n = \frac{1}{n'}, \quad p = \frac{1}{p'}, \quad q = \frac{1}{q'};$$

and from (8)

$$\frac{1}{m'n'}\left[\frac{q'-p'}{p'q'}\right] = \frac{1}{p'q'}\left[\frac{n'-m'}{m'n'}\right]; \qquad (21)$$

$$\therefore \qquad q'-p' = n'-m'. \qquad (22)$$

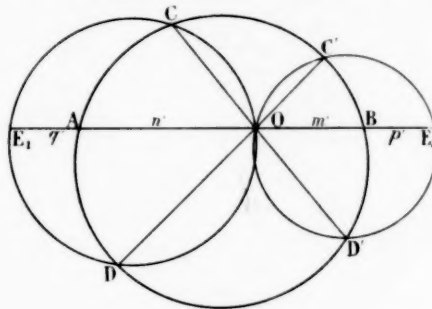∴ The difference of the segments intercepted by the two circles is equal

Fig. 4.

to the difference of the segments intercepted by the given circle.

Applying the same method to Fig. 2, using a similar notation, and substituting in (8) and (12), give

$$n'-m' = q'-p' = s'-r'. \qquad (23)$$

∴ If circles be passed through COC' and DOD', COD and $C'OD'$, respectively, the differences of the segments determined by these *two pairs* of circles on any transversal is always the same, and equal to the difference of the intercepts of the given circle.

It is obvious that if the same process be applied to Fig. 3, and the results substituted in (18) and (20) we should have

$$m'+n' = q'+p' \quad \text{and} \quad p'+q' = r'+s', \qquad (24)$$

respectively.

∴ When O is without the given circle (regarding the algebraic signs as before) the sums of the segments made by the two pairs of opposite circles are always equal whether the transversal cuts the given circle or not.

Now when O becomes the middle point of AB,

$$n' = m', \quad q' = p', \quad s' = r', \qquad (24a)$$

from (23), and also

$$n = m, \quad q = p, \quad s = r.$$

∴ When the segments intercepted by the given circle are equal, both pairs of opposite chords and both pairs of opposite circles described upon them intercept equal segments also.

Although these results have been proved by inversion it is clear that they are perfectly general, and hence we may drop the primes from m and n.

So far we have discussed the two parts of this problem separately. Having found these separate relations, we shall now proceed to consider the complete solution.

Since the equations connecting the segments made by opposite chords and the given circle, and also those connecting segments made by the given
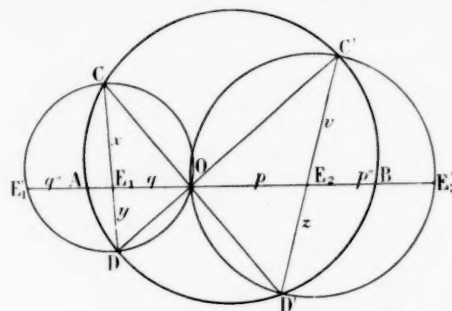

Fig. 5.

circle and a pair of circles upon these chords, have been shown to be identical under all possible conditions, it is sufficient now to consider only the simplest case, and we shall use the first.

Let Fig. 5 be the same as Fig. 1 with circles drawn through COD and $C''OD'$ respectively ; let the notation be the same, and in addition let $OE_2' = p'$, $OE_1' = q'$, $E_2E_2' = p''$, $E_1E_1' = q''$; then

$$\frac{xy}{vz} = \frac{qq''}{pp''}. \tag{25}$$

On making the figures for all the different cases the truth of this equation also is at once perceived to be general.

Now from the first two fractions of (6) (and this equation is also found to be common to the proof of all the four cases), and from (25), we obtain

$$\frac{q^2}{p^2} = \frac{qq''}{pp''}, \tag{26}$$

whence $\dfrac{q}{p} = \dfrac{q''}{p''}$, or

$$\therefore \qquad\qquad p : p'' :: q : q''. \tag{27}$$

By composition, remembering that $p + p'' = p'$ and $q + q'' = q'$, we get

$$p : p' :: q : q'. \tag{28}$$

From these two proportions the following general statements may now be made :—

Segments intercepted by two circles are divided proportionally by the chords upon which the circles are drawn.

Segments made by two opposite chords and circles described upon these chords are also proportional.

When O is on the given circle, say at A, the circle COD reduces to a point, and q, q' are both equal to zero, and from (28) $\dfrac{p}{p'}$ takes the form $\dfrac{0}{0}$ and may have any value between zero and unity.

Now let

$$\frac{p}{p'} = \frac{q}{q'} = k ; \tag{29}$$

then $p = kp'$, $q = kq'$; and these values substituted in (8) give

$$mn\,(p' - q') = kp'q'\,(m - n) ; \tag{30}$$

hence by (22)

$$mn = kp'q' , \tag{31}$$

and by (29)

$$\frac{mn}{p'q'} = \frac{p}{p'} = \frac{q}{q'} ; \tag{32}$$

$\therefore$

$$mn = pq' = p'q . \tag{33}$$

$\therefore$ Segments intercepted by a chord and the circle described upon the opposite chord are reciprocally proportional to the segments intercepted by the given circle.

Now in Fig. 5 draw the chords CC'', DD', and upon these also describe circles passing through O. We thus have an inscribed quadrilateral with a circle described upon every side and passing through the intersection of the diagonals as shown in Fig. 6.

Let the notation for the first pair of lines and circles be the same, and also let $OE_4 = r$, $OE_1 = r'$, $OE' = s$, $OE_4' = s'$. Then, since (33) is true under all conditions, we have

$$mn = rs' = r's . \tag{34}$$

However, this can be shown independently by a process just like the one used from (26) to (33), using (12) and (23) instead of (8) and (22).

In the same way using (18), (19), and (24) we can show that (33), (34) are true when O is without the given circle.

Now combining (33) and (34) gives

$$mn = pq' = p'q = rs' = r's . \tag{35}$$

Since the product mn is constant it follows that this equation is also true when O is outside and the transversal does not cut the given circle. Thus the

formula in (35) is perfectly general and expresses the desired fundamental relation.
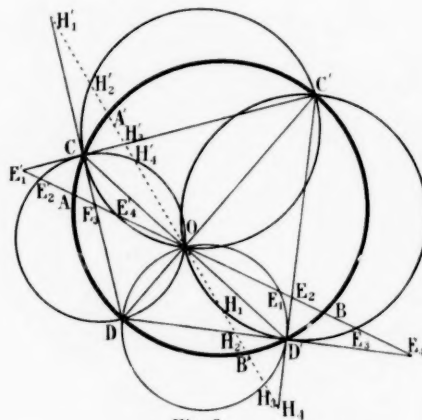

Fig. 6.

Next, to consider its consequences.   Every right line through O cuts each of the sides of the quadrilateral, each of the four circles through O in one other point and the given circle in two points (when O is within it), so that eight of the ten factors in (35) are always real, and when one passes through infinity another must pass through zero.   Therefore when the transversal is parallel to a side of the quadrilateral it must be tangent to the circle described on the opposite side,* and when one point of intersection changes its direction from O another changes also, and thus two lines change signs simultaneously.

As a particular case, when O is the middle point of AB it is readily seen from $(24a)$ that (35) becomes

$$m^2 = n^2 = pp' = qq' = rr' = ss' . \qquad (36)$$

Restoring the values of the factors in (35) as shown on Fig. 6 we have

$$OA . OB = OE_1 . OE_1' = OE_2 . OE_2' = OE_3 . OE_3' = OE_4 . OE_4' . \quad (37)$$

It is now seen that the points A, B; E_1, E_1'; E_2, E_2'; E_3, E_3'; E_4, E_4', form a system in involution of which the point O is the centre.   Also that one pair of conjugate points lies on the given circle, and of every other pair one point lies on a side and its conjugate lies on the circle described on the opposite side.

Since $OA . OB$ is constant it follows that when O is outside the given circle the locus of the foci is a circle whose centre is O and whose radius is a mean proportional between the segments of either diagonal.

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* This result is easily proved by means of arcs, and is cited simply as a check for this solution of the given problem.

As a result of this investigation the following general theorem can now be enunciated :—

THEOREM I. If a quadrilateral is inscribed in a given circle, and on each of its sides circles are described passing through the intersection of the diagonals, and any right line is also drawn through the intersection of the diagonals :

1. The product of the segments intercepted on this right line by the given circle, and the product of the segments intercepted by two opposite sides of the quadrilateral, are proportional to their respective $\begin{cases} \text{differences.} \\ \text{sums.} \end{cases}$ The same is true of the segments intercepted by the two pairs of opposite sides. See (9), (13), (20).

2. The $\begin{cases} \text{difference} \\ \text{sum} \end{cases}$ of the segments intercepted by the given circle is equal to the $\begin{cases} \text{difference} \\ \text{sum} \end{cases}$ of the segments intercepted by two circles described upon opposite sides of the quadrilateral. The same is true of the segments intercepted by the two pairs of opposite circles. See (22), (23), (24).

3. The segments intercepted by two opposite circles are divided proportionally by the sides upon which these circles are described, and are also proportional to the segments intercepted by the same sides. See (27), (28).

4. This right line (or transversal) is cut by the five circles and the four sides of the quadrilateral in ten points in involution of which the intersection of the diagonals is the centre. See (37).

5. When the intersection of the diagonals is outside of the given circle the locus of the foci* of the involution is a circle with the same centre as the involution and whose radius is a mean proportional between the segments of either diagonal.

6. When the segments intercepted by the given circle are equal, the segments intercepted by opposite sides, and the segments intercepted by opposite circles, are equal also ; and the segments made by the given circle are a mean proportional between the segments made by any side and the circle described upon that side. See (10), (24a), (36).

7. On every transversal (under conditions of 5) are determined five harmonic ranges, every range including the two foci ; for any two conjugate points of an involution and the two foci form a harmonicrange (Smith's Conics, p. 59).

Let any other line $A'OB'$ (Fig. 6) be drawn giving the pairs of conjugate points, H_1, H_1' ; H_2, H_2' ; H_3, H_3', etc.; then since

$$OA \cdot OB = OA' \cdot OB',$$

$$\therefore \qquad OE_1 \cdot OE_1' = OH_2 \cdot OH_2' = OE_4 \cdot OE_4', \text{ etc.}$$

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* When the point is within there are no foci, for conjugate points are then on opposite sides of the centre.

8. ∴ Any pair of conjugate points on one transversal and any pair on any other transversal are concyclic.

Before extending this investigation any further it is necessary to call special attention to the fact that the term "quadrilateral" as here used does not mean the "complete quadrilateral" of modern geometry, which is considered as having three diagonals each of which is divided harmonically by the other two. On the contrary, any four points on the given circle are here supposed to determine in general three different inscribed quadrilaterals, as the three different pairs of intersecting lines are taken successively for diagonals. Thus the sides of these three inscribed quadrilaterals are all chords (not produced) of the given circle, and there are three points of "intersection of diagonals,"* one within and two without the given circle.

Inasmuch as all of the preceding formulas have been shown to be true under all possible conditions it is only necessary to say that Theorem I can be applied to all of these quadrilaterals.

With the aid of this theorem we can now prove very neatly a few additional theorems relating to concurrence and collinearity, and also discover some interesting relations connecting these three quadrilaterals and their systems of circles.

The reader is now referred to Fig. 7, which is necessarily a somewhat complicated diagram representing the three quadrilaterals with their systems of circles, and also the loci of the foci of two involutions, in all fifteen circles. The given circle and the system of quadrilaterals are drawn in heavy lines, and the four systems of circles in light lines ; the two focal loci are represented by dotted lines.

The three D-points are O_1, O_2, O_3. In the discussion of this figure it seems desirable to note, in the first place, that for each exterior D-point, in addition to the system of circles through that point, there are two circles belonging to each of the other systems which are coaxial with the given circle and having this D-point for their radical centre. Therefore these circles also cut any transversal drawn through this D-point in points which belong to the same involution, conjugate points lying on the same circle. A transversal may cut one, two, or three of these, so that there may now be as many as sixteen points and can not be fewer than eight in any such involution.

Bearing in mind, as has been stated, that if a point in any of these involutions lies on a side of the quadrilateral, its conjugate lies on the circle

* Hereafter, instead of "intersection of diagonals," we shall write "D-point." And four circles described on the sides of one quadrilateral and through its D-point, will be called a "system" of circles or "system." The two quadrilaterals whose D-points are without the given circle may be called "crossed quadrilaterals," since one pair of their opposite sides intersect within the given circle. Circles described upon opposite sides will be called "opposite circles."

described on the opposite side and *vice versa*, it is clear that the intersection
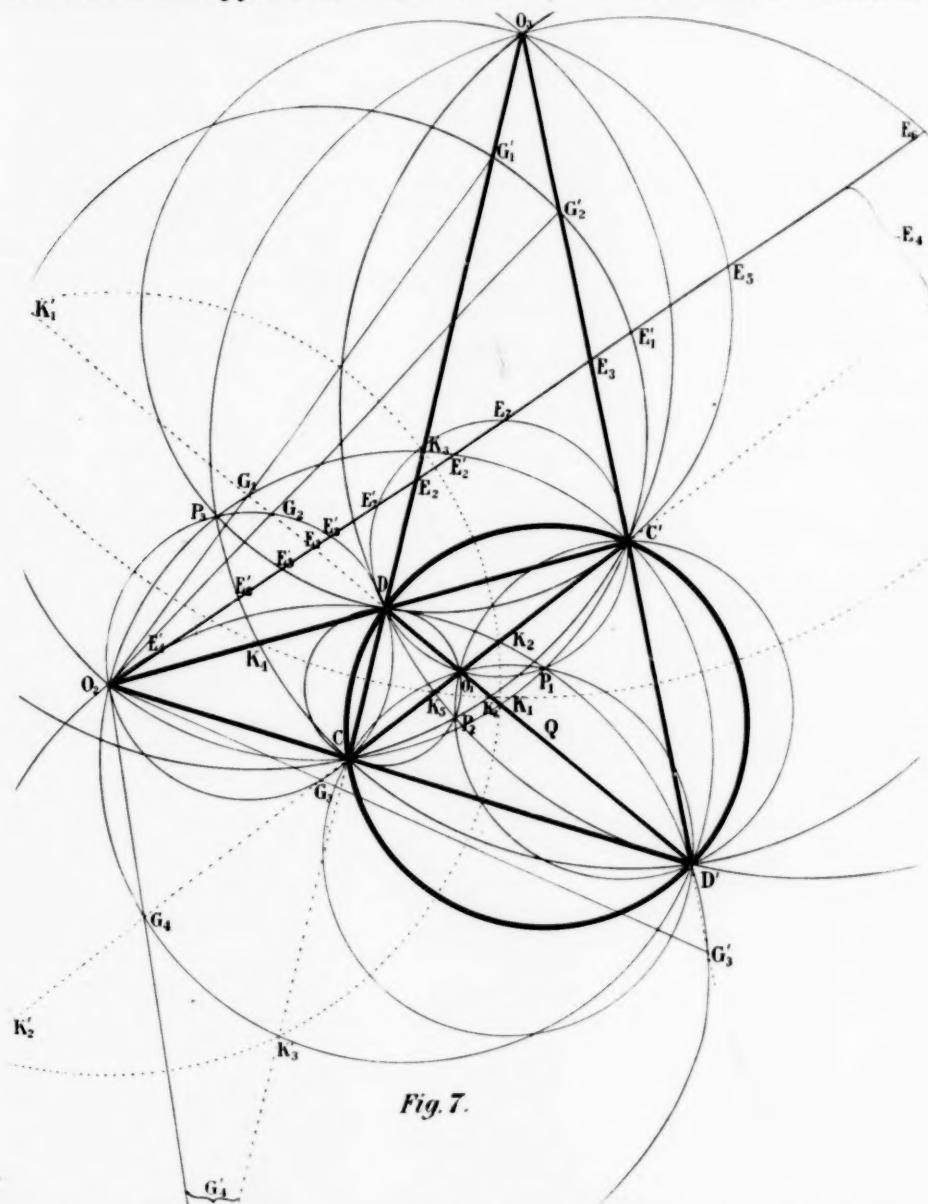

Fig. 7.

of a circle with the side opposite to that upon which it is described becomes two coincident conjugate points and is therefore a focus.

(a) ∴ A circle described on a side of a crossed quadrilateral meets the

opposite side in a point whose distance from the D-point is a mean proportional between the segments of either diagonal.

For each exterior D-point there are six such points, some of which are shown at K_1, K_1', K_2, K_2', K_3, K_3' for O_2, and K_4, K_5, K_6, for O_3.

If, in an involution, two points which are not conjugates coincide, the points which are conjugate to these respectively must also coincide. Thus, to the intersection of a side and an adjacent circle there corresponds the intersection of a circle and an adjacent side.

(b) $\therefore$ Two circles of the same system described upon adjacent sides meet the other two adjacent sides in two points which are collinear with the D-point of the system.

There are four such lines for each system, but only those for O_2 are drawn, viz. $O_2 G_1 G_1'$, $O_2 G_2 G_2'$, $O_2 G_3 G_3'$, $O_2 G_4 G_4'$.

Also, to the intersection of two opposite sides there corresponds the intersection of the circles upon those sides. This last point must be considered further. Let any right line be drawn through O_2 meeting the sides (counting from O_2) in E_1, E_2, E_3, E_4, and the circles on the respective opposite sides in E_1', E_2', E_3', E_4'; let this line also meet the circle $O_3 C'' D$ in E_5, E_5' and the circle $O_3 C D'$ in E_6, E_6'; conjugate points being designated by the same subscripts. Now suppose this line to revolve about O_2 until it passes through O_3, then E_2, E_3, E_5, E_6 coincide in O_3, and therefore their respective conjugates E_2', E_3', E_5', E_6' must coincide in some point as P_3. By supposing this line to revolve in the opposite direction till it passes through O_1 we can show that four different circles meet in a common point at P_1. This last statement will now be proved in a different way in order to secure an additional inference.

Using the former notation, and adhering to the convention that the segments intercepted by the given circle and also those intercepted by circles upon opposite sides shall have like signs when they extend in the same direction from the D-point, it will be remembered that (22) becomes general. It also follows that when the segments intercepted by two opposite circles coincide they must have the same sign, and (22) takes the form

$$m - n = p' + q'. \tag{22'}$$

Therefore, when a transversal passes through the intersection of two opposite circles, and hence also through the intersection of two opposite sides, we must have $p' = + q'$. Then by substituting in (22') we have

$$m - n = 2p', \text{ for } O_2; \tag{38}$$

and

$$m + n = 2p', \text{ for } O_3. \tag{39}$$

Thus in both cases the intersection of the opposite circles bisects the chord

intercepted by the given circle, and therefore all four circles meet in a common point. In precisely the same way it can be shown that the point P_2 is the middle of the chord intercepted on a line through O_3O_1, and also common to four circles.

(c) We thus have three other points* besides the three D-points through each of which four circles pass (no circle passing through more than one), and lying on the lines joining the D-points.

A second general theorem may now be stated:

THEOREM II. If a quadrilateral is inscribed in a given circle and on each of its sides circles are described passing through the D-point:

1. In a crossed quadrilateral three of these circles meet their respective opposite sides in points whose distance from the D-point is the mean proportional between the segments of either diagonal.   (a)

2. Two circles upon adjacent sides meet the other two adjacent sides in points which are collinear with the D-point.   (b)

3. Two circles described upon one pair of opposite sides and passing through the intersection of the other pair, and two circles described upon the second pair passing through the intersection of the first pair, all meet in a common point on the right line joining the intersections of the opposite sides, and bisect the chord of the given circle drawn through this point and the D-point.

For the sake of perspicuity in the diagram, let us now transfer the three D-points and these three P-points to Fig. 8. In this figure the triangle determined by the three D-points is drawn in heavy continuous lines and two sides produced to P-points; and the three P-points are joined with heavy dotted lines.

Now remembering that O_1, O_2, O_3 are each the centre of an involution in which each of the other two is the conjugate of a P-point, we have

$$O_1P_2 \cdot O_1O_3 = O_1P_1 \cdot O_1O_2, \quad O_2O_1 \cdot O_2P_1 = O_2P_3 \cdot O_2O_3,$$

$$\text{and } O_3O_1 \cdot O_3P_2 = O_3P_3 \cdot O_3O_2; \quad (40)$$

hence the following groups of points are concyclic:

$$O_2,\ O_3,\ P_1,\ P_2;\ O_3,\ P_1,\ O_1,\ P_3;\ O_2,\ P_3,\ O_1,\ P_2.$$

Draw these three circles in continuous lines and circumscribe a dotted circle about the triangle $O_1O_2O_3$; also draw the straight lines O_2P_2, O_1P_3, and O_3P_1. We now have a system of quadrilaterals having a side in the common line O_2O_3 and inscribed in each of these continuous circles and three circles through each of the points O_1, O_2, O_3.

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* These three points are marked P_1, P_2, P_3, and when necessary to refer to them in a general way we shall write "P-point."

Consider the first of these quadrilaterals $O_2O_3P_1P_2$ with its interior D-point O_1. It is easily seen that each of the three circles through O_1 is described upon a side of this quadrilateral, and therefore any transversal
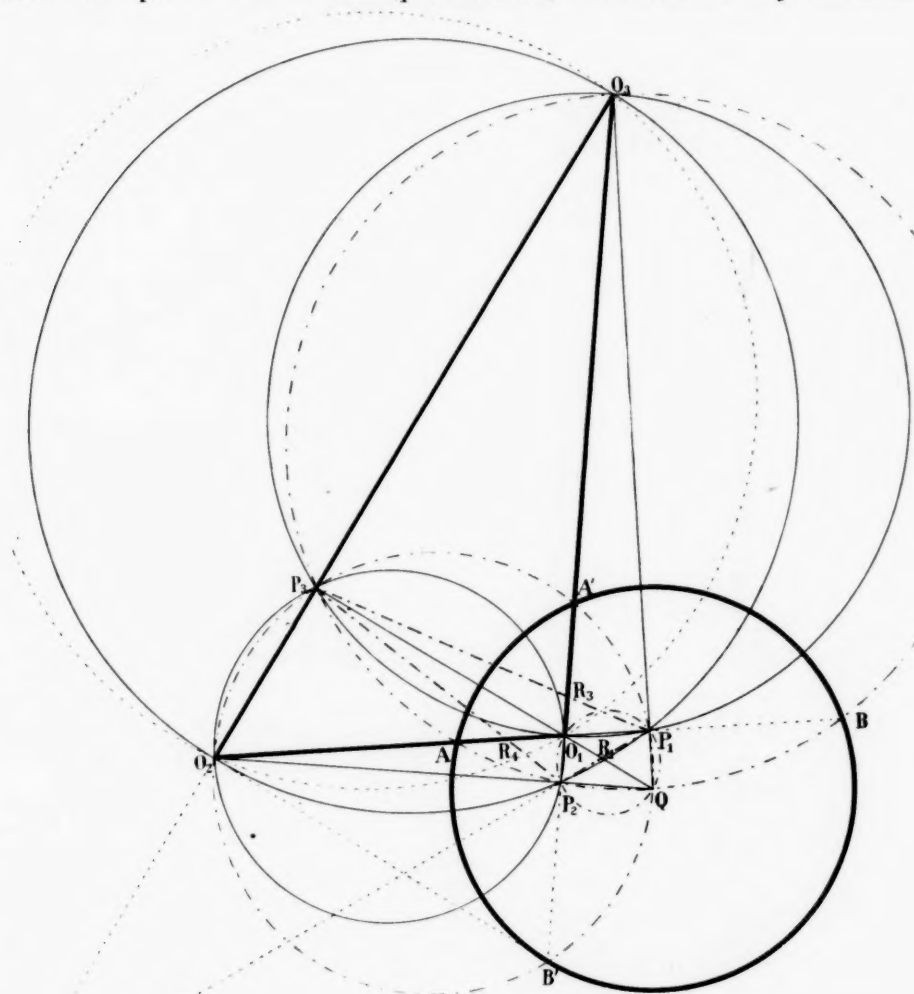

Fig. 8.

through O_1 is cut in involution by this system of circles and lines, and from the first equation of (40) it is evident that these points belong to the same involution we have been considering. The same thing can be shown to be true at the points O_2 and O_3, so that we may now add eight more points to every involution.

Furthermore it is readily perceived that

$$\angle\, O_2P_1P_2 = \angle\, O_2P_1P_3 = \angle\, O_3\,,\quad \angle\, O_3P_2P_1 = \angle\, O_3P_2P_3 = \angle\, O_2\,.$$

$\therefore$ The triangles $O_1P_1P_2,\, O_2P_1P_3,\, O_3P_2P_3$ are each similar to $O_1O_2O_3$; the sides of the triangle $P_1P_2P_3$ are the *antiparallels* of the sides of the triangle $O_1O_2O_3$; and the sides of the triangle $O_1O_2O_3$ bisect the angles (exterior angle at P_3) of the triangle $P_1P_2P_3$.

Hence O_1 is the centre of the circle inscribed in the triangle $P_1P_2P_3$, and therefore O_1P_3 bisects the $\angle\, P_1P_3P_2$, and O_1P_3 is perpendicular to O_2O_3.

$\therefore$ O_3O_1 is the diameter of the circle $O_3P_3O_1P_1$,

$\qquad O_2O_1$ "   "   "   " $O_2P_3O_1P_2$,

and $\qquad O_2O_3$ "   "   "   " $O_3P_1P_2O_2$.

$\therefore$ A circle described on a line joining two D-points as a diameter passes through the two P-points that do not lie on that line.

As the lines O_1P_3, and $O_2P_3O_3$ bisect respectively the interior and exterior angle of the triangle $P_1P_2P_3$, the lines P_3P_1, P_3O_1, P_3P_2, P_3O_2 form a harmonic pencil intercepting the harmonic ranges $P_1R_1P_2R_2,\, P_1O_1R_4O_2,\, P_2O_1R_3O_3$.

For the same reason there are harmonic pencils at P_1 and P_2 intercepting the same range $O_3P_3O_2R_2$.

One ray of each of these pencils passes through each of the D-points. Also any line through a D-point is a transversal of all these pencils, save when it coincides with a ray.

$\therefore$ Every right line through a D-point is not only cut in involution, but is also cut in three harmonic ranges to which this D-point is common.

It has been proved that O_3P_1, O_1P_3, O_2P_2 are perpendicular to O_2O_1, O_2O_3, O_1O_3, respectively, and also, in (38), (39), that P_1, P_2 are the middle points of chords of the given circle which lie in these lines O_2O_1, O_3O_1.

$\therefore$ The three altitudes of the triangle $O_1O_2O_3$ meet in Q, which is the centre of the given circle.

Next describe a circle upon O_2Q as a diameter. This circle, drawn in dots and dashes, passes through P_1, P_3, since $\angle\, O_2P_1Q = \angle\, O_2P_3Q =$ Rt. $\angle$. Also a circle on O_3Q as a diameter passes through P_2, P_3; and a circle on O_1Q as a diameter passes through P_1, P_2.

$\therefore$ As these circles are each described upon a side of the quadrilaterals now under consideration, it is evident that they and their corresponding sides give points in the same involution and that the centre of the given circle is a point in the involution on the lines drawn through it, its conjugates being respectively P_1, P_2, P_3.

Let us now consider the quadrilateral $P_3O_1O_3P_1P_3$. The circle upon O_2Q as a diameter, being also described on P_1P_3, intersects the opposite side O_3O_1 in two points, as A', B', which lie on the locus of the foci, since both are coin-

cident conjugate points.   It also cuts the given circle in two points which are
the points of contact of tangents through O_2; but these points of contact are
also coincident conjugate points, and hence lie on the locus of the foci.   There-
fore the two circles and the locus of the foci meet the line O_3O_1 in the same
two points.

And since O_2A', O_2B' are tangents to the given circle, O_2 is the pole of
O_3O_1 with respect to this circle.   In the same way it can be shown that O_3 is
the pole of O_2O_1, and therefore O_1 is the pole of O_2O_3.

∴ The triangle of the D-points is self-polar with respect to the given
circle; which is a well known theorem.

Again, it is easily seen that Q is a common D-point for all the quadri-
laterals that are inscribed in the circles which are described upon the sides of
the self-polar triangle as diameters, and is therefore the radical centre of these
circles.   Therefore, any right line through Q is cut in involution by the sides
of the self-polar triangle and by all these circles (save the one circumscribing
$O_1O_2O_3$).   In these involutions if a point lies on a circle through Q and a vertex
of the self-polar triangle, its conjugate lies on the side opposite this vertex,
other circles cutting in conjugate points.

Again, from the right triangle QO_2B it is evident that

$$QP_2 . QO_2 = \overline{QB'}^2.$$

∴ The given circle is the locus of the foci of these involutions and cuts
orthogonally the circles described upon the sides of the self-polar triangle.

These last results merit special attention and will be put in the form of a
general theorem.

THEOREM III. If a triangle is self-polar with respect to a given circle, and
circles are described on its sides as diameters, and also three other circles are
drawn having for their diameters the right lines joining the centre of the given
circle (intersection of altitudes) to the vertices of the self-polar triangle :

1. Every straight line drawn through the centre of the given circle is cut
by these six circles and the sides of the self-polar triangle in an involution of
which the centre of the given circle is the centre.

2. The given circle is the locus of the foci of the involution determined
upon this variable line, and also cuts orthogonally the three circles described
upon the sides of the self-polar triangle.

3. Every diameter of the given circle is divided harmonically, (1) by
each of the circles upon the sides of the self-polar triangle (provided it cuts
the diameter), (2) by each side of the self-polar triangle and its respective circle
which passes through the opposite vertex of the triangle and the centre of the
given circle.

[TO BE CONCLUDED.]

CONTENTS.

ANNALS OF MATHEMATICS.